CONFLUENT HYPERGEOMETRIC FUNCTIONS

H.A. Lauwerier.

1. Introduction.

The confluent hypergeometric functions are defined as the solutions of the following equation:

$$\frac{d^2 W}{d z^2} + \left\{ A + \frac{B}{z} + \frac{C}{z^2} \right\} W = 0$$

or in standardized notation

$$\frac{d^2 W}{d z^2} + \left\{ -\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2} \right\} W = 0 \tag{1}$$

Putting $W = e^{-\frac{z}{2}} z^{m+\frac{1}{2}} v$, (1) may be transformed into

$$z \frac{d^2 v}{d z^2} - (z - \rho) \frac{dv}{dz} - \alpha v = 0 \tag{2}$$

with $\rho = 2m+1$ and $\alpha = \frac{1}{2} + m-k$.

Equation (2) is satisfied by the series

$$v = {}_1F_1(\alpha, \rho, z) = 1 + \frac{\alpha}{\rho} z + \frac{\alpha(\alpha+1)}{\rho(\rho+1)} \frac{z^2}{2!} + \dots \tag{3}$$

Another solution is found to be

$$v = z^{1-\rho} \, {}_1F_1(1+\alpha-\rho, 2-\rho, z) \tag{4}$$

For non integral values of ρ we get two independent series solutions but difficulties occur when ρ is integral.

The solution of (1) obtained from (3) will be denoted by $M_{k,m}(z)$:

$$M_{k,m}(z) = e^{-\frac{z}{2}} z^{m+\frac{1}{2}} \, {}_1F_1(\tfrac{1}{2} + m-k, 2m+1, z) \tag{5}$$

Another solution is obtained when substituting m by -m :

$$M_{k,-m}(z) = e^{-\frac{z}{2}} z^{-m+\frac{1}{2}} \, {}_1F_1(\tfrac{1}{2}-m-k, -2m+1, z) \tag{6}$$

The solution (6) can also be obtained from (4).

When 2m is not an integer (5) and (6) form a fundamental system since the singularities in the origin are of a different nature.

Since these functions depend on three variables they are practically beyond the reach of tabulation in general. If a function of one variable takes a page to tabulate, one of two variables will take a book and one of three variables an ordinary sized room of bookshelves.

Consequently the theory of this function is mainly a matter of general propositions with detailed application to a few special cases. In the following pages it will be shown where in mathematical physics C.H. functions occur and to which problems, in particular asymptotic expansions, they give lead. The concerning asymptotic expansions will be treated in detail.

2. Integral representation.

Integral representations can be obtained by Laplace's method : Putting in (2)

$$v = \int_L e^{\lambda z} \varphi(\lambda)\, d\lambda \tag{7}$$

we find for φ the conditions:

$$\left.(\lambda-\lambda^2)\varphi e^{\lambda z}\right|_L = 0$$

$$-(\lambda^2\varphi)' + (\lambda\varphi)' + \rho\lambda\varphi - \alpha\varphi = 0 \tag{8}$$

or

$$\frac{\varphi'}{\varphi} = \frac{-1+\alpha+(2-\rho)\lambda}{\lambda-\lambda^2}$$

which is satisfied by $\varphi = \dfrac{(\lambda-1)^{-\alpha+\rho-1}}{\lambda^{1-\alpha}} = \dfrac{(\lambda-1)^{k+m-\frac{1}{2}}}{\lambda^{k-m+\frac{1}{2}}}$

Thus the solutions of (2) can be brought in the form:

$$v = C\int_L e^{\lambda z}\frac{(1-\lambda)^{k+m-\frac{1}{2}}}{\lambda^{k-m+\frac{1}{2}}}\, d\lambda \tag{9}$$

C a certain constant.

From (8) it follows that L ~~can~~ may be a non overlapping contour starting from infinity where $\frac{\pi}{2} < \arg \lambda z \leq \pi$ encircling one or two singularities (the origin and the point $\lambda = 1$) and returning to infinity in a suited direction.

In (9) the following useful transformations may be performed:

$$\lambda = \frac{t}{z}$$

$$\text{or} \quad \lambda = \frac{1+u}{2}$$

From (3) we obtain as a standard solution of (1) the so called function of Whittaker:

$$W_{k,m}(z) = \frac{1}{2\pi i}\,\Gamma(\tfrac{1}{2}+k-m)\, e^{-\frac{z}{2}}\, z^k \int^{0^+} e^t\, t^{-k+m-\frac{1}{2}}\left(1-\frac{t}{z}\right)^{k+m-\frac{1}{2}} d \quad (1.$$

The advantage of the use of this function in stead of the $M_{k,m}(z)$ function is due to the fact that $W_{k,m}(z)$ has a meaning for all values of k,m,z in particular for integral values of 2m. Moreover Whittakers function admits a very simple asymptotic representation.

It may be remarked that, if not otherwise started, the complex powers are understood in the convential way e.g. $t^{\alpha} = e^{\alpha \ln t}$; arg t = 0 for real positive t, for other t values along the contour t^{α} is defined by analytic continuation.

(10) can also be brought in the following more symmetrical form:

$$W_{k,m}(z) = \frac{1}{\pi i} \Gamma(\tfrac{1}{2} + k - m)(\tfrac{z}{4})^{m+\frac{1}{2}} \int^{(-1)^+} e^{\frac{uz}{2}} (\tfrac{1-u}{1+u})^{k} \frac{du}{(1-u^2)^{-m+\frac{1}{2}}} . \qquad (11)$$

Formulae (10) and (11) become meaningless when $-\frac{1}{2} + k - m$ is a negative integer. For these critical values another representation may be used, obtained from (10) by a simple transformation:

$$W_{k,m}(z) = \frac{e^{-\frac{z}{2}} z^{k}}{\Gamma(\frac{1}{2} - k + m)} \int_0^{\infty} t^{-k-\frac{1}{2}+m} . (1 + \tfrac{t}{z})^{k-\frac{1}{2}+m} e^{-t} dt \qquad (12),$$

$\mathrm{Re}\,(-\frac{1}{2} + k - m) < 0$.

Further on the following representation of the M function will be proved:

$$M_{k,m}(z) = \frac{\Gamma(2m+1)}{\pi i} (\tfrac{4}{z})^{m-\frac{1}{2}} \int^{(\pm 1)^+} e^{\frac{uz}{2}} (\tfrac{u-1}{u+1})^{k} \frac{du}{(u^2-1)^{m+\frac{1}{2}}} \qquad (13)$$

The contour is a non overlapping curve starting from infinity encircling both singularities ± 1 in a positive direction and returning to infinity.

<u>Proof</u>;

$$\text{right hand side} = \frac{\Gamma(2m+1)}{2\pi i} z^{\frac{1}{2}-m} e^{-\frac{z}{2}} \int^{(0,1)^+} e^{vz} \frac{(v-1)^{k-m-\frac{1}{2}}}{v^{k+m+\frac{1}{2}}} dv =$$

$$= \frac{\Gamma(2m+1)}{2\pi i} z^{\frac{1}{2}-m} e^{-\frac{z}{2}} \int e^{vz} v^{-1-2m} (1 - \tfrac{1}{v})^{k-m-\frac{1}{2}} dv =$$

$$= \Gamma(2m+1) z^{\frac{1}{2}-m} e^{-\frac{z}{2}} \sum_0^{\infty} (-1)^j \binom{k-m-\frac{1}{2}}{j} \frac{1}{2\pi i} \int e^{vz} v^{-2m-j-1} dv =$$

$$= \frac{\Gamma(2m+1)}{\Gamma(\frac{1}{2} - k + m)} e^{-\frac{z}{2}} z^{\frac{1}{2}-m} \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} - k + m + j)}{j!} \frac{z^{2m+j}}{\Gamma(j+2m+1)} =$$

$$= z^{m+\frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1(\tfrac{1}{2} - k + m, 2m+1, z) = M_{k,m}(z)$$

according to (5)

Par. 3 <u>Asymptotic expansion of</u> $W_{k,m}(z)$

From formula (10) it is easy to derive an asymptotic series for

$W_{k,m}(z)$. In order to get an estimation for the error the following lemma, due to Jacobi will be proved:

<u>lemma</u> $$(1+z)^m = \sum_0^n \binom{m}{j} z^j + \binom{m}{n+1}(1+z)^m \int_0^z \frac{dt^{n+1}}{(1+t)^{m+1}}$$

<u>Proof</u>;

$$R_n = -\frac{m-n}{m}\binom{m}{n}(1+z)^m \int_0^z t^n \, d\,\frac{1}{(1+t)^m} =$$

$$= -\frac{m-n}{m}\binom{m}{n} z^n + \frac{m-n}{m}\binom{m}{n}(1+z)^m \int_0^z (1+t) \frac{dt^n}{(1+t)^{m+1}} =$$

$$= -\frac{m-n}{m}\binom{m}{n} z^n + \frac{m-n}{m} R_{n-1} + \frac{n}{m} R_n$$

hence we have the reduction formula

$$R_n = R_{n-1} - \binom{m}{n} z^n$$

From this and

$$R_o = (1+z)^m - 1$$

the lemma easily follows.

Applying this lemma on formula (10) we get:

$$\int^{0^+} e^t \, t^{-k+m-\frac{1}{2}} \left(1-\frac{t}{z}\right)^{k+m-\frac{1}{2}} dt = \sum_0^\infty (-1)^j \binom{k+m-\frac{1}{2}}{j} z^{-j} \,.$$

$$\cdot \int^{0^+} e^t \, t^{-k+m-\frac{1}{2}+j} \, dt + \int^{0^+} e^t \, t^{-k+m-\frac{1}{2}} R_n\left(k+m-\tfrac{1}{2}, -\tfrac{t}{z}\right) dt$$

$$\therefore \; W_{k,m}(z) = e^{-\frac{z}{2}} z^k \left\{ \sum_0^n (-1)^j \frac{\Gamma(k+\frac{1}{2}+m)\,\Gamma(k+\frac{1}{2}-m)}{\Gamma(k-j+\frac{1}{2}+m)\,\Gamma(k-j+\frac{1}{2}-m)} \frac{z^{-j}}{j\,!} + \text{error} \right\}; \tag{15}$$

(15) may be written also as

$$W_{k,m}(z) = e^{-\frac{z}{2}} z^k \left\{ \sum_0^n \frac{\prod^j \left\{ m^2 - (k-l+\frac{1}{2})^2 \right\}}{j\,!\, z^j} + \text{error} \right\} \tag{16}$$

Using formula (12) for the error, we find, supposing sufficiently large:

$$\text{error} = \frac{\Gamma(\frac{1}{2}+k+m)}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(k+m-\frac{1}{2}-n)\,n!} \int_0^\infty e^{-t} \, t^{-k+m-\frac{1}{2}} \left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} \varphi(t)\,dt$$

where $$\varphi = \int_0^{t/z} \frac{u^n \, du}{(1+u)^{k+m+\frac{1}{2}}}$$

Denoting the coefficient of z^{-j} in the asymptotic expansion of $W_{km}(z)e^{z/2}z^{-k}$ by A_j, the error may be expressed also in the following form:

$$\text{error} = \frac{(n+1)A_{n+1}}{z^{n+1}\,\Gamma(n+1+1/2-k+m)} \int_0^\infty e^{-t}\, t^{-k+m-1/2}\left(1+\frac{t}{z}\right)^{k+m-1/2} \int_0^t \frac{u^n\,du\,dt}{\left(1+\frac{u}{z}\right)^{k+m+1/2}}$$

Supposing $|\arg z| < \pi$ We have for large values of $|z|$

$$|\text{error}| = O\left(\frac{1}{|z|^{n+1}}\right)$$

as the reader will prove without difficulty.

The following special cases will be considered:

a) $\operatorname{Re} z = x \geqslant 0$ $\quad$ k, m real; $\quad k+m > -1/2$

For $|z| > 1$ we have

$$\left|\int_0^t \frac{u^n du}{\left(1+\frac{u}{z}\right)^{k+m+1/2}}\right| \leqq \int_0^t \frac{u^n du}{\left|1+\frac{u}{z}\right|^{k+m+1/2}} \leqq \int_0^t u^n du = \frac{t^{n+1}}{n+1}$$

$$\left|1+\frac{t}{z}\right|^{k+m-1/2} \leqq (1+t)^{k+m+1/2}$$

hence (n sufficiently large to secure convergence):

$$|\text{error}| \leqq \frac{A_{n+1}}{\Gamma(n+1+1/2-k+m)} \int_0^\infty e^{-t}\, t^{-k+m+n+1/2}\,(1+t)^{k+m+1/2}\,dt \quad (18)$$

It is not possible to express the integral in (18) in elementary functions such as gamma functions. But however a simple approximation can be derived.

Starting from the function

$$f(a,b) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t}\, t^{a-1}\,(1+t)^b dt \qquad \begin{matrix} a > 1 \\ b > -1 \end{matrix}$$

We get the following transformations

$$f = \frac{1}{2\pi i} \int_0^\infty e^{-t}\, t^{a-1}(1+t)^b \int_L e^w\, w^{-a}\, dw\,dt$$

where L is a loop round the negative real axis.

Further:

$$f = \frac{1}{2\pi i} \int_0^\infty (1+t)^b dt \int_L e^{-t(1-w)}\, w^{-a}\, dw.$$

The contour L can be transformed into a vertical line:

$$w = k+ui \qquad \begin{cases} 0 < k < 1 \\ -\infty < u < +\infty \end{cases} \qquad \begin{matrix} k \text{ constant} \\ u \text{ variable} \end{matrix}$$

Performing the substitution it is found:

$$f = \frac{1}{2\pi} \int_0^\infty (1+t)^b dt \int_{-\infty}^\infty e^{-(1-k)t+uti}\, (k+ui)^{-a} du$$

$$|f| \leqq \frac{1}{2\pi} \int_0^\infty e^{-(1-k)t}\,(1+t)^b dt \int_{-\infty}^\infty (k^2+u^2)^{-1/2a} du =$$

$$= \frac{k^{-\frac{1}{2}(a-1)}}{2\pi} \int_1^\infty e^{-(1-k)(v-1)} \quad v^b dv \int_0^\infty \frac{w^{-1/2}}{(w+1)^{\frac{1}{2}a}} dw \leq$$

$$= \frac{e^{1-k}\,\Gamma(1+b)}{2\pi k^{\frac{1}{2}(a-1)}(1-k)^{1+b}} \quad B\,(1/2,\ \tfrac{1}{2}a-1/2) =$$

$$= \frac{e^{1-k}}{2\pi^{1/2} k^{\frac{1}{2}(a-1)}(1-k)^{1+b}} \quad \frac{\Gamma(\frac{1}{2}a-1/2)\ \Gamma(1+b)}{\Gamma(\frac{1}{2}a)}$$

Choosing $k = 1/2$ we find:

$$f(a,b) \leq 2^{\frac{1}{2}a+b} \sqrt{\frac{e}{2\pi}} \quad \frac{\Gamma(\frac{1}{2}a-1/2)\ \Gamma(1+b)}{\Gamma(\frac{1}{2}a)} \tag{19}$$

For high values of a the quotient

$\frac{\Gamma(\frac{1}{2}a-1/2)}{\Gamma(\frac{1}{2}a)}$ can be approximated as follows:

$$\ln\frac{\Gamma(x-1/2)}{\Gamma(x)} = (x-1)\ln(x-1/2)-(x-1/2)\ \ln x+1/2+$$

$$+\tfrac{1}{2}\int_0^\infty \frac{P_1(t)}{(x+t)(x+t-1/2)}dt$$

where $P_1(t) = 1/2-t+[t]$

Using the second mean value theorem we set:

$$\ln\frac{\Gamma(x-1/2)}{\Gamma(x)} = (x-1)\ \ln(1-\frac{1}{2x})-\tfrac{1}{2}\ln x+1/2+\frac{1}{2x(x-1/2)}\int_0^\beta P_1(t)dt =$$

$$= -(1/2+\frac{1}{8x}+\frac{1}{24x^2}\ \ldots) + (\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{24x^3}\ \ldots) -$$

$$- \tfrac{1}{2}\ln x+1/2+\frac{\Theta}{16x(x-1/2)} \qquad 0<\Theta<1$$

or for $x \geqslant 1$

$$\ln\frac{\Gamma(x-1/2)}{\Gamma(x)} \leq -\frac{1}{2}\ln x+\frac{3}{8x}+\frac{1}{8x^2(1-\frac{1}{2x})}+\frac{1}{16x^2(1-\frac{1}{2x})} -$$

$$-\frac{1}{24x^2} \leq -\frac{1}{2}\ln x+\frac{3}{8x}+\frac{1}{3x^2}$$

hence:

$$\frac{\Gamma(x-1/2)}{\Gamma(x)} \leq \frac{e^{\frac{3}{8x}+\frac{1}{3x^2}}}{\sqrt{x}} \leq \frac{1}{\sqrt{x-1}} \tag{20}$$

For high values of a (19) may be replaced by

$$f(a,b) \leq \frac{2^{\frac{1}{2}a+b}}{\sqrt{\pi a}}\, e^{\frac{1}{2}+\frac{3}{4a}+\frac{4}{3a^2}}\ \Gamma(1+b) \tag{21}$$

For the remainder in the asymtotic development (case a) we get the approximation:

$$m+k+1/2 = \alpha \geqslant 0$$
$$m-k+1/2 = \beta$$

$$|\text{error}| \leq 2^\alpha\ \Gamma(1+\alpha)\sqrt{\frac{e}{\pi}}\ \frac{2^{1/2(n+1+\beta)}}{(n+1+\beta)^{1/2}}\ \exp\left\{\frac{3}{4(n+1+\beta)}+\right.$$

$$+ \frac{4}{3(n+1+\beta)^2} \Big\} \frac{|A_{n+1}|}{|z|^{n+1}} \qquad (22)$$

valid for $n \geqq 1-\beta$

b) $\operatorname{Re} z \geqq 0$ $\qquad$ k,m real, $k+m \leqq -1/2$

Now we have

$$\left| \frac{(1+\frac{u}{z})^{-(k+m+1/2)}}{(1+\frac{t}{z})^{-(k+m-1/2)}} \right| = \left| \frac{1+\frac{u}{z}}{1+\frac{t}{z}} \right|^{-(k+m+1/2)} \leqq 1$$

Hence we find the very simple expression:

$$|\text{error}| \leqq \frac{|A_{n+1}|}{|z|^{n+1}} \qquad (23)$$

the ideal result for an asymptotic series.

Remark 1 As will be seen in the following section the functions $W_{k,m}(z)$ and $W_{k,-m}(z)$ are identical.

This means for the remainder an important extension of the range of validity:

(22) is valid in the range $k+|m|+1/2 \geqq 0$

α = least positive value of $k\pm m+1/2$

(23) is valid in the range $k+1/2 \leqq |m|$

Remark 2.(*)

Second proof of the Jacobi formula (lemma of §3):

The Taylor expansion of $(1+z)^a$ with the remainder in integral form is as follows:

$$f(z) \equiv (1+z)^a = \sum_0^n \frac{z^k}{k!} f^{(k)}(0) + \frac{z^{n+1}}{n!}\int_0^1 (1-u)^n f^{(n+1)}(uz)\,du =$$

$$= \sum_0^n \frac{z^k}{k!} f^{(k)}(0) + \frac{a(a-1)\dots(a-n)}{n!} z^{n+1} \int_0^1 (1-u)^n (1+uz)^{a-n-1} du$$

Putting $u = \frac{1-v}{zv+1}$ it is found

$$\int_0^1 (1-u)^n (1+uz)^{a-n-1} du = (1+z)^a \int_0^1 v^n (1+vz)^{-a-1} dv$$

which proves Javobi's lemma.

x x x x x

(*) Cf. Whittaker and Watson. Modern Analysis, 5.41 and ch.5 ex.6.

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§ 4. Consequences.

$W_{k,m}(z)$ and $W_{-k,m}(z\,e^{\pi i})$ form a standard set of solutions of the basic differential equation. Their independency follows from the fact that they admit different asymptotic development

The functions $W_{k,-m}(z)$ and $W_{-k,-m}(z\,e^{\pi i})$ are solutions as well and each of them thus may be expressed as a linear combination of $W_{k,m}(z)$ and $W_{-k,m}(z\,e^{\pi i})$. However, since $W_{k,m}(z)$ and $W_{k,-m}(z)$ have exactly the same asymptotic representation, it follows

$$W_{k,-m}(z) = W_{k,m}(z) \tag{19}$$

a result, difficult to prove by means of the integral representation.

Returning to the original differential equation (1)

$$\frac{d^2\,W}{d\,z^2} + \left\{ -\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2} \right\} \cdot W = 0 \tag{1}$$

there are three important solutions:

$$\left\{\begin{aligned} W^{(1)}_{k,m}(z) &= \frac{1}{\pi i}\,\Gamma(\tfrac{1}{2}+k+m)\,\left(\frac{z}{4}\right)^{-m+\frac{1}{2}} \int^{(-1)^+} e^{\frac{uz}{2}} \left(\frac{1-u}{1+u}\right)^k \frac{du}{(1-u^2)^{m+\frac{1}{2}}} && (20)\\ W^{(2)}_{k,m}(z) &= \frac{1}{\pi i}\,(\tfrac{1}{2}-k+m)\,\left(\frac{z}{4}\right)^{-m+\frac{1}{2}} \int^{(+1)^+} e^{\frac{uz}{2}} \left(\frac{1-u}{1+u}\right)^k \frac{du}{(1-u^2)^{m+\frac{1}{2}}} && (21)\\ M_{k,m}(z) &= \frac{1}{\pi i}\,(1+2\,m)\,\left(\frac{z}{4}\right)^{-m+\frac{1}{2}} \int^{(\pm 1)^+} e^{\frac{uz}{2}} \left(\frac{u-1}{u+1}\right)^k \frac{du}{(u^2-1)^{m+\frac{1}{2}}} && (22) \end{aligned}\right.$$

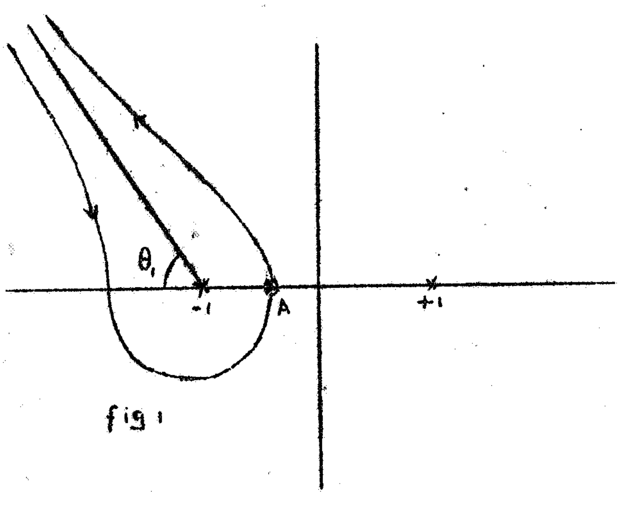


fig 1

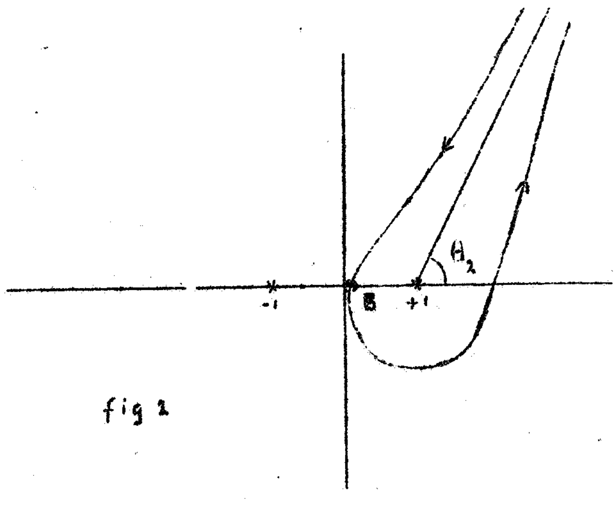


fig 2

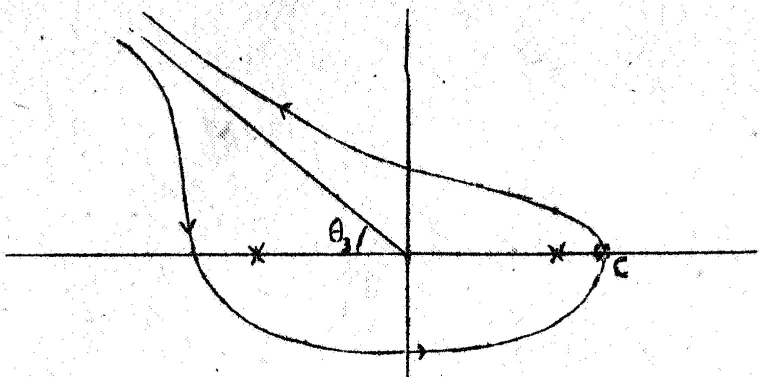


fig 3

1) The contour in (20), a loop round the singularity -1 in a certain direction, is represented in fig. 1. To assure one-valuedness in A on t real axis between -1 and $+1$ we take $\arg(1-u) = \arg(1+u) = 0$.

θ_1 may take any value in the interval

$$-\pi < \theta_1 < +\pi$$

Formula (20) thus is valid in the region

$$-\frac{\pi}{2} + \theta_1 < \arg z < \frac{\pi}{2} + \theta_1$$

Hence $W^{(1)}_{k,m}(z)$ is uniquely defined in the whole region

$$-3\frac{\pi}{2} < \arg z < +3\frac{\pi}{2}$$

2) Analogously for the contour in formula (21), represented in fig.2 we have in B on the real axis

$$\arg(1-u) = \arg(1+u) = 0$$

θ_2 is bounded by

$$-\pi < \theta_2 < \pi$$

Formula (21) is valid for all z for which

$$-3\frac{\pi}{2} + \theta_2 < \arg z < -\frac{\pi}{2} + \theta_2$$

Consequently $W^{(2)}_{k,m}(z)$ is uniquely defined in the region:

$$-5\frac{\pi}{2} < \arg z < \frac{\pi}{2}$$

3) The contour of $M_{k,m}(z)$ is represented in fig. 3.

In C on the real axis we take again

$$\arg(1-u) = \arg(1+u) = 0$$

θ_3 is lying in the interval

$$-\pi < \theta_3 < +\pi$$

Formula (22) is valid for all z for which:

$$-\frac{\pi}{2}+\ \theta_3 < \arg z < +\frac{\pi}{2}+\theta_3$$

Hence $M_{k,m}(z)$ is defined in the region

$$-3\frac{\pi}{2} < \arg z < +3\frac{\pi}{2}$$

§ 5. At first a summary of the properties of $W^{(1)}_{k,m}$ and $W^{(2)}_{k,m}$ will be given:

$$\left\{\begin{aligned} &W^{(1)}_{k,m}(z) = W_{k,m}(z) && (23)\\ &W^{(2)}_{k,m}(z) = e^{(m+\frac{1}{2})\pi i}\, W^{(1)}_{-k,m}(z\, e^{\pi i}) && (24)\end{aligned}\right.$$

$$\left\{\begin{aligned} &W^{(1)}_{k,-m}(z) = W^{(1)}_{k,m}(z) && (25)\\ &W^{(2)}_{k,-m}(z) = e^{-2m\pi i}\, W^{(2)}_{k,m}(z) && (26)\end{aligned}\right.$$

$$M_{k,m}(z) = e^{(k-m-\frac{1}{2})\pi i}\left\{\frac{\Gamma(2m+1)}{\Gamma(k+m+\frac{1}{2})}\, W^{(1)}_{k,m}(z) + \frac{\Gamma(2m+1)}{\Gamma(-k+m+\frac{1}{2})}\, W^{(2)}_{k,m}(z)\right\} \qquad (27)$$

$$\left\{\begin{aligned} &M_{k,m}(e^{2l\pi i}z) = e^{l(2m+1)\pi i}\, M_{k,m}(z) && (28)\\ & && l \text{ an integer}\\ &M_{k,m}(e^{\pi i}z) = e^{(m+\frac{1}{2})\pi i}\, M_{-k,m}(z) && (29)\end{aligned}\right.$$

$$W^{(1)}_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(-k-m+\frac{1}{2})}\, M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(-k+m+\frac{1}{2})}\, M_{k,-m}(z) \qquad (30)$$

$$W^{(2)}_{k,m}(z) = \frac{\Gamma(-2m)\, e^{(2m+1)\pi i}}{\Gamma(k-m+\frac{1}{2})}\, M_{k,m}(z) - \frac{\Gamma(2m)}{\Gamma(k+m+\frac{1}{2})}\, M_{k,-m}(z) \qquad (31)$$

$$W^{(1)}_{k,m}(z) = e^{-\frac{z}{2}}\, z^{k}\left\{1 + \sum_{1}^{N}\frac{\{m^2-(k-\frac{1}{2})^2\}\cdot\{m^2-(k-\frac{3}{2})^2\}-\{m^2-(k-n+\frac{1}{2})^2\}}{n!\; z^n} + O(z^{-N-1})\right\} \qquad (32)$$

valid in the region $-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$

$$W^{(2)}_{k,m}(z) = e^{\frac{z}{2}+(k+m+\frac{1}{2})\pi i}\, z^{-k}\left\{1 + \sum_{1}^{N}(-1)^n\frac{\{m^2-(k+\frac{1}{2})^2\}\{m^2-(k+\frac{3}{2})^2\}\ldots\{m^2-(k+n-\frac{1}{2})^2\}}{h!\; z^n.} + O(z^{-N-1})\right\} \qquad (33)$$

valid in the region $-\frac{5\pi}{2} < \arg z < \frac{\pi}{2}$

$$M_{k,m}(z)=\frac{\Gamma(2m+1)}{\Gamma(-k+m+\frac{1}{2})}\; e^{\frac{z}{2}}\; z^{-k}\left\{\sum_{0}^{N}\frac{\{(k+\frac{1}{2})^2-m^2\}\ldots\{(k+n-\frac{1}{2})^2-m^2\}}{n!\;z^n}+\right.$$

$$\left.+\,O(z^{-N-1})\right\}+e^{(k-m-\frac{1}{2})\pi i}\,e^{-\frac{z}{2}}\,z^{k}.$$

$$\cdot\left\{\sum_{0}^{N}\frac{\{m^2-(k-\frac{1}{2})^2\}\ldots\{m^2-(k-n+\frac{1}{2})^2\}}{n!\;z^n}+O(z^{-N-1})\right\} \tag{34}$$

valid in the region $-3\frac{\pi}{2}<\arg z<\frac{\pi}{2}$.

§6. Proofs of the formulas in §5.

The relation (23) is obvious, (24) can be proved by a simple transformation of the variable of integration:

Supposing $-\frac{5\pi}{2}<\arg z<\frac{\pi}{2}$ we have

$$W^{(1)}_{-k,m}(z\,e^{\pi i})=\frac{1}{\pi i}\;\Gamma(-k+m+\tfrac{1}{2})e^{(-m+\frac{1}{2})\pi i}\;(\tfrac{z}{4})^{-m+\frac{1}{2}}\int^{(-1)^+} e^{\frac{1}{2}uze^{\pi i}}\,.$$

$$\cdot\left(\frac{1-u}{1+u}\right)^{-k}\frac{du}{(1-u^2)^{m+\frac{1}{2}}}=-\frac{1}{\pi i}\Gamma(-k+m+\tfrac{1}{2})\;e^{(-m+\frac{1}{2})\pi i}\;(\tfrac{z}{4})^{-m+\frac{1}{2}}\int^{(+1)^+}e^{\frac{1}{2}vz}$$

$$\cdot\left(\frac{1-v}{1+v}\right)^{k}\frac{d}{(1-v^2)^{m+\frac{1}{2}}}=e^{-(m+\frac{1}{2})\pi i}\;W^{(2)}_{k,m}\;(z)$$

Formulae (25) and (26) are simple consequences from (23) and (24).

The proof of (27) is simple too. The contour of (22) is equivalent to the combination of the contours of (20) and (21), further on $u-1=e^{\pi i}\,(1-u)$.

Since $M_{k,m}(z)=z^{m+\frac{1}{2}}$ analytic function, the relations (28) and (29) are evident.

Replacing in (27) m by -m and making use of (25) and (26) we may eliminate either $W^{(1)}$ or $W^{(2)}$ and we get formulae expressing the W functions as linear combinations of $M_{k,m}(z)$ and $M_{k,-m}(z)$. These formulae (30) and (31) define $W^{(1)}$ and $W^{(2)}$ for unrestricted phase of z, and it is easy matter to express let us say $W^{(1)}_{k,m}(z\,e^{2l\pi i})$ where l is any integer as a linear combination of $M_{k,m}(z)$ and $M_{k,-m}(z)$ or of $W^{(1)}_{k,m}(z)$ and $W^{(2)}_{k,m}(z)$. The formulae obtained in this way may be useful in order to get asymptotic representations valid for unrestricted phase of z. Hence it is sufficient to get asymptotic relations in a strip of 2π for arg z. The fact however that the asymptotic relations obtained viz. (32) and (33) are valid in the extended range of 3π gives rises to the phenomenon of Stokes, i.e. there are regions where two different asymptotic representations are valid at the same time.

§ 7. The asymptotic representations:

Starting from

$$W^{(1)}_{k,m}(z)= \frac{1}{2\pi i}\Gamma(k-m+\tfrac{1}{2})\ e^{-\frac{z}{2}}\ z^k \int^{0^+} e^t\ t^{-k+m-\frac{1}{2}}\ (1-\tfrac{t}{z})^{k+m-\frac{1}{2}}\, dt \qquad (35)$$

we follow the same way as devised by Watson for the Hankelfunctions which are indeed special cases of Whittakers functions.

The path of integration in (35 is a loop round the line

$$-t = \tau e^{\beta i} \qquad \tau \geq 0,\quad |\beta| < \tfrac{\pi}{2}$$

(35) is valid for $\beta - \pi < \arg z < \beta + \pi$

The development of $(1-\frac{t}{z})^\lambda$ $\lambda = k + m - \frac{1}{2}$ into a series as a special case of the Taylor series in the following form:

$$f(a+h)= \sum_0^n \frac{h^j}{j!}\ f^{(j)}(a) + \frac{h^{n+1}}{n!}\int_0^1 (1-u)^n\ f^{(n+1)}(a+uh)du$$

is

$$(1-\tfrac{t}{z})^\lambda = \sum_0^n (-1)^n\ \frac{\lambda(\lambda-1)\ldots(\lambda-n+1)}{n!}\ \frac{t^n}{z^n} + \frac{\lambda(\lambda-1)\ldots(\lambda-n)}{n!}\cdot$$
$$\cdot\left(\tfrac{t}{z}\right)^{n+1}\cdot\int_0^1 (1-u)^n\ (1-\tfrac{ut}{z})^{\lambda-n-1}\ du \qquad (36)$$

It is convenient to take n so large that

$$\mathrm{Re}(k \pm m - n - \tfrac{3}{2}) \leq 0$$

Then we choose any positive angle δ which satisfies the inequalities:

$$|\beta| \leq \tfrac{\pi}{2} - \delta \qquad |\beta - \arg z| \leq \pi - \delta$$

From this it follows

$$-\frac{3\pi}{2} + 2\delta \leq \arg z \leq \frac{3\pi}{2} - 2\delta$$

Hence the minimum value of $1 - \frac{tu}{z}$ is the distance of the point +1 to the line $\tau e^{\delta i}$ $(-\infty < \tau < +\infty)$ or

$$\left|1 - \frac{tu}{z}\right| \geq \sin\delta$$

Further

$$\left|\arg(1 - \tfrac{tu}{z})\right| < \pi$$

Applying this it is found

$$\left|(1 - \frac{ut}{z})^{\lambda-n-1}\right| \quad e^{\pi|\mathrm{Im}\lambda|} \qquad (\sin\delta)^{\mathrm{Re}(\lambda-n-1)} = A_{n+1}$$

For the remainder in (36) we get the approximation:

The contour of the integral in the remainder of the asymptotic development of $e^{\frac{z}{t}}$ z^{-k} $W^{(1)}_{k,m}(z)$ may be transformed into twice the straight line $-t = \tau e^{\beta i}$ and we may write:

$$|R| \leqslant \frac{|\Gamma(k-m+\tfrac{1}{2}).\ \lambda(\lambda-1)\ldots(\lambda-n)|}{\pi\, n!\ \ |z|^{n+1}} \left|\sin(-k+m+n+\tfrac{3}{2})\pi\right| \ .$$

$$A_{n+i} \int_0^\infty e^{-t\cos\beta}\ \tau^{Re(-k+m+n+\frac{1}{2})}\, d\tau\ .\int_0^1 (1-u)^n\, du$$

or
$$|R| \leqslant \frac{A_{n+1}}{(n+1)!\ \pi\ |z|^{n+1}} \left|\sin(k-m+\tfrac{1}{2})\ \Gamma(k-m+\tfrac{1}{2}).\ \lambda(\lambda-1)\ldots(\lambda-n)\right|$$

$$\frac{\Gamma\{Re(-k+m+n+\frac{3}{2})\}}{(\cos\beta)^{Re(-k+m+n+\frac{3}{2})}} \qquad (37)$$

Indeed it follows from (37) for $|\arg z| < \frac{3\pi}{2}$ and for all k,m values, negative integer values for $k-m+\frac{1}{2}$ included, that

$$|R| = O(|z|^{-n-1})$$

This proves (32). Using (24) we get (33).
Combining (32) and (33) and using (27) the asymptotic representation of $M_{k,m}(z)$ is found, valid in a phase range 2π.

CONFL ENT HYPERGEOMETRIC FUNCTIONS

§5. Some special cases of Whittaker functions

Some important special cases are given here without proof:

1) cylinderfunctions

$$\begin{cases} H_m^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-(m+1/2)\frac{\pi i}{2}} W_{0,m}\left(2e^{\frac{\pi i}{2}} z\right) \\ H_m^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{(m+1/2)\frac{\pi i}{2}} W_{0,m}\left(2e^{\frac{\pi i}{2}} z\right) \end{cases}$$

or $$H_m^{(k)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-(m+1/2)\frac{\pi i}{2}} W_{0,m}^{(k)}\left(2e^{-\frac{\pi i}{2}} z\right)$$

$$K_m(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,m}(2z)$$

$$J_m(z) = 2^{-2m-1/2} z^{-1/2} e^{(m+1/2)\frac{\pi i}{2}} \frac{M_{0,m}\left(2e^{-\frac{\pi i}{2}} z\right)}{\Gamma(m+1)}$$

$$I_m(z) = \frac{2^{-2m}(2z)^{-1/2}}{\Gamma(m+1)} M_{0,m}(2z)$$

2) incomplete Γ-function

$$\Gamma(2m) - \int_0^z t^{2m-1} e^{-t} dt = z^{m-1/2} e^{-1/2 \cdot z} W_{m-1/2,m}(z)$$

3) Error function

$$\int_z^\infty e^{-t^2} dt = \frac{1}{2} z^{-1/2} e^{-1/2 z^2} W_{-1/4,1/4}(z^2)$$

4) Laguerre functions

$$L_n^{(a)}(z) = \frac{\Gamma(a+n+1)}{\Gamma(a+1)\ \Gamma(n+1)} z^{-\frac{a+1}{2}} e^{\frac{z}{2}} M_{\frac{a+1}{2}+n,\ \frac{a}{2}}(z)$$

5) Hermite functions or Parabolic cylinderfunctions

$$D_n(z) = 2^{\frac{n}{2}+\frac{1}{4}} z^{-1/2} W_{\frac{n}{2}+\frac{1}{4},\ -\frac{1}{4}}\left(\frac{z^2}{2}\right)$$

$$He_n(z) = e^{\frac{z^2}{4}} D_n(z)$$

§6. A physical problem

An interesting physical problem which needs the use of confluent hypergeometric functions in the theory of heat effects in capillary flow (Cf. a paper of Dr H.C. Brinkman to appear in applied scientific research) will be treated below.

The velocity of flow in a capillary is given by Poiseuille's law:

Asympt. Ontw.

$$v = \frac{R^2 - r^2}{4\eta} \quad \frac{dp}{dz}$$

where

R radius of the capillary while the z-axis coincides with the axis of the capillary; r distance to the axis.

η viscosity

p pressure assumed to vary linearly with z.

The heat of friction generated in the energy done on an element of volume by the normal and shearing stresses:

$$\varphi = \frac{r^2}{4\eta} \quad \left(\frac{dp}{dz}\right)^2$$

For the stationnary state the temperature distribution is determined by a differential equation expressing the heat balance for an element of volume:

$$-\lambda\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right\} + v c \frac{\partial T}{\partial z} = \frac{r^2}{4\eta}\left(\frac{dp}{dz}\right)^2 \qquad (1)$$

where

λ is the heat conductivity and

c is the specific heat per unit of volume

The first term of (1) is related to the heat transport by conduction, the second term to that by convection.

The heat conduction in the z-direction is supposed so small that this may be neglected.

For sufficiently large v this is physically permitted.

The boundary conditions may be;

1. the walls of the capillary are kept at constant temperature: T=0 for r=R.

or 2. the walls of the capillary have zero heat conductivity: $\frac{\partial T}{\partial r} = 0$, for r=R.

Moreover it is assumed that the fluid is introduced into the capillary at zero temperature:

T=0 for z=0.

Choosing new variables, the whole problem may be restated mathematically as follows:

$$\frac{1}{r}\ \frac{\partial}{\partial r}\ \left(r\ \frac{\partial w}{\partial r} \right) = 4(1-r^2)\ \frac{\partial w}{\partial t} - 16r^2 \qquad (2)$$

$$\begin{array}{llll} t=0 & w=0 & & \\ r=1 & w=0 & \text{or} & \dfrac{\partial w}{\partial r} = 0 \end{array}$$

It is easily seen that $w=C-r^4$ satisfies (2) and for suitable C the boundary conditions.

Introducing a new dependent variable $w'=w+(c-r^4)$ it is found for w' the following equation:

$$\frac{1}{r}\ \frac{\partial}{\partial r}\left(r\ \frac{\partial w}{\partial r}\right) = 4(1-r^2)\ \frac{\partial w}{\partial t}$$

$$\begin{array}{lll} r=1 & w=0 \quad \text{or} \quad \dfrac{\partial w}{\partial r}=0 & \\ r=0 & w=\text{finite} & \quad (3) \\ z=0 & w=\text{given function of } r & \end{array}$$

Of course the apparently more general equation:

$$\frac{1}{r}\ \frac{\partial}{\partial r}\left(r\ \frac{\partial w}{\partial r}\right) = (A-Br^2)\ \frac{\partial w}{\partial t}$$

may be reduced easily to the form in (3).

The general solution may be represented by

$$w = \sum a_n\ e^{-\omega_n^2 t}\ \varphi_n(r)$$

The eigenfunction $\varphi_n(r)$ satisfies the ordinary second-order differential equation:

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + 4\omega^2\ (1-r^2)\varphi \ = 0 \qquad (4)$$

Introducing new variables defined by

$$x = 2\ \omega\ r^2$$

$$\varphi = \frac{1}{r}\ \psi$$

(4) is transformed into

$$\psi'' + \left\{\frac{1}{4x^2} + \frac{\omega}{2x} - \frac{1}{4}\right\}\psi \ = 0 \qquad (5)$$

the well-known equation o f the Whittaker functions:

$$W_{\frac{\omega}{2},o}(x) \quad \text{and} \quad W_{-\frac{\omega}{2},o}(-x)$$

To avoid a singularity in the origin we have to choose the M-function.

Hence the eigonfunctions are:

$$\varphi = \frac{1}{r}\ M_{\frac{\omega}{2},o}\ (2\omega\, r^2)$$

or $$\varphi = (2\ \omega)^{1/2}\ e^{-\ \omega\ r^2}\ {}_1F_1\left(\frac{1-\omega}{2},\ 1,\ 2\ \omega\ r^2\right)$$

or $$= (2\ \omega)^{1/2}\ e^{-\omega\, r^2}\sum_0^{\infty}\ \frac{(1-\omega)(3-\omega)\ldots(2n-1-\omega)}{n!\ n!}\ .\ (\omega r^2)^n \qquad (6)$$

In case 1, which will be considered only, the eigenvalue equation is $\varphi(1) = 0$ or $M_{\frac{\omega}{2},o}(2\omega) = 0$ (7)

From the series representation of (7) only the first eigenvalues can be calculated and therefore it is advisable to derive an asymptotic expansion for the functions

$$M_{\frac{\omega}{2};o}(2\omega)$$

and

$$M_{\frac{\omega}{2},o}(2\omega r^2)$$

It may be remarked that from a well-known theorem of Laguerre it follows that (7) admits only real positive zeros.

CONFLUENT HYPERGEOMETRIC FUNCTIONS

9. Asymptotic development of $M_{\frac{\omega}{2},0}(2\omega)$.

The following formula has been proved

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{(2\omega)^{\frac{1}{2}}}{2\pi i} \int^{(\pm 1)} e^{\omega f(u)} g(u)\,du \qquad (8)$$

where

$$f(u) = u + \tfrac{1}{2} \ln \frac{u-1}{u+1}$$

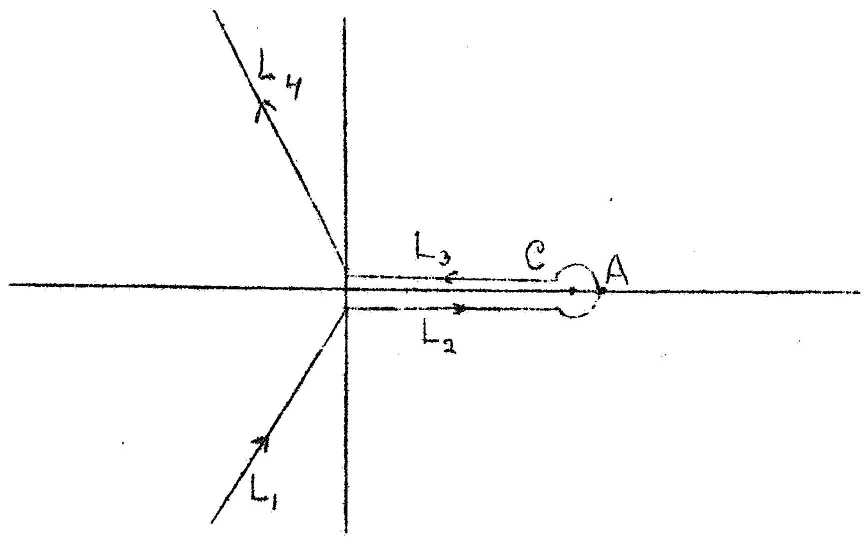
$$g(u) = (u^2-1)^{-\frac{1}{2}}$$

Only those parts of the contour lying in the vicinity of the saddlepoints, i.e. where $f'(u)=0$, contribute essentially to the value of the integral.

Here
$$f'(u) = \frac{u^2}{u^2-1}$$

hence the origin is the only saddlepoint, and of the second order.

In accordance to this fact the path of integration can be chosen as follows:



L_1 :	$u = t \exp -\frac{2}{3}\pi i$	$0 \leq t < \infty$
L_2 :	$u = t$, $\arg(u-1) = -\pi$	$0 \leq t \leq 1-\delta$
C :	$u - 1 = \delta \exp \varphi i$	$-\pi \leq \varphi \leq \pi$
L_3 :	$u = t$, $\arg(u-1) = \pi$	$0 \leq t \leq 1-\delta$
L_4 :	$u = t \exp \frac{2}{3}\pi i$	$0 \leq t < \infty$

It is easily seen that the contribution of the δ-circle round

+ 1 tends to zero as $\delta \to 0$.

Now the following lemma can be proved

lemma

$$(1-z^2)^{-\frac{1}{2}} \; e^{\omega\left\{z+\frac{1}{2}\ln\frac{1-z}{1+z}\right\}} = e^{\frac{\omega}{3}z^3}\left\{\sum_0^n c_k z^k + R_n(z)\right\} \tag{9}$$

$$R_n(z) \to 0 \quad \text{for} \quad n \to \infty \;,\; |z| < 1.$$

The coefficients c_k satisfy the recurrency relation

$$\begin{cases} (k+1)c_{k+1} = k\,c_{k-1} - \omega\,c_{k-4} \\ \qquad c_0 = 1 \end{cases}$$

c_k is a polynomial of degree $\left[\frac{k}{5}\right]$ in ω.

Proof. Supposing $|z| < 1$ for $F(z) = \sum_0^\infty c_k z^k$ the following differential equation may be derived

$$(1-z^2)\,\frac{dF}{dz} = (z - \omega z^4)\,F \tag{11}$$

Substituting again the series expression and equating powers z^k it is found

$$(k+1)\,c_{k+1} = k\,c_{k-1} - \omega\,c_{k-4}$$

From $F(0) = 1$ it follows $c_0 = 1$.

In particular we have

$$\begin{array}{lll} c_0 = 1 & c_5 = -\frac{1}{5}\,\omega & c_{10} = \frac{63}{256} + \frac{1}{50}\,\omega^2 \\ c_1 = 0 & c_6 = \frac{5}{16} & \\ c_2 = \frac{1}{2} & c_7 = -\frac{17}{70}\,\omega & \\ c_3 = 0 & c_8 = \frac{35}{128} & \\ c_4 = \frac{3}{8} & c_9 = -\frac{649}{2520}\,\omega & \end{array}$$

The remainder R_n will be estimated later on.

Now the asymptotic development may be easily obtained:

$$(2\omega)^{-\frac{1}{2}}\, M_{\frac{\omega}{2},0}(2\omega) = \frac{1}{2\pi i}\int_{L_1+L_2+L_3+L_4} e^{u\omega}\left(\frac{u-1}{u+1}\right)^{\frac{\omega}{2}} \frac{du}{(u^2-1)^{\frac{1}{2}}} =$$

$$= \frac{1}{\pi}\, J_m \int_{L_1+L_2} = \frac{1}{\pi}\, I_m \left\{ e^{-\frac{\omega-1}{2}\pi i} \int_{L_1+L_2} e^{-\frac{\omega}{3}u^3} \left\{ \sum_0^n c_k u_k + R_n \right\} du \right\} =$$

$$= \frac{1}{\pi} \sum_{0}^{n} c_k \, I_m \Big\{ - e^{\frac{\omega-1}{2}\pi i} \int_0^\infty e^{-\frac{\omega}{3} t^3} \left(t e^{\frac{2\pi i}{3}}\right)^k d\left(t e^{\frac{2\pi i}{3}}\right) +$$

$$+ e^{\frac{\omega-1}{2}\pi i} \int_0^1 e^{-\frac{\omega}{3} t^3} t^k \, dt + e^{-\frac{\omega-1}{2}\pi i} \int_{L_1+L_2} e^{-\frac{\omega}{3} u^3} R_n \, du \Big\} =$$

$$= \frac{1}{\pi} \sum_{0}^{n} c_k \sin\left(\frac{\omega-1}{2} + \frac{2k+2}{3}\right)\pi \int_0^\infty e^{-\frac{\omega}{3} t^3} t^k \, dt - \frac{1}{\pi} \sum_{0}^{n} c_k \sin \frac{\omega-1}{2}\pi \,\cdot$$

$$\cdot \int_0^\infty e^{-\frac{\omega}{3} t^3} t^k \, dt \; + \; \text{remainder}$$

$$\text{remainder} = \frac{1}{\pi} \sum_{0}^{n} c_k \sin \frac{\omega-1}{2}\pi \int_1^\infty e^{-\frac{\omega}{3} t^3} t^k \, dt + \frac{1}{\pi} I_m \, e^{\frac{\omega-1}{2}\pi i} \int_{L_1+L_2} e^{-\frac{\omega}{3} u^3} R_n \, du$$

Hence

$$(2\omega)^{-\frac{1}{2}} M_{\frac{\omega}{2},0}(2\omega) = \frac{2}{3\pi} \sum_{0}^{n} c_k \sin \frac{k+1}{3}\pi \cos\left(\frac{\omega}{2} + \frac{k}{3} - \frac{1}{6}\right)\pi \,\cdot$$

$$\cdot \Gamma\left(\frac{k+1}{3}\right)\left(\frac{3}{\omega}\right)^{\frac{k+1}{3}} + \qquad (12)$$

$$+ \text{ Remainder}$$

$$\left|\text{Remainder}\right| \leqq A \, e^{-\frac{\omega}{3}} + \frac{1}{\pi} \int_0^\infty e^{-\frac{\omega}{3} t^3} \left| R_n\left(t \, e^{\frac{2\pi i}{3}}\right) \right| dt \; +$$

$$+ \frac{1}{\pi} \int_0^1 e^{-\frac{\omega}{3} t^3} \left| R_n(t) \right| dt$$

where A is bounded for $\omega \to \infty$.

<u>Estimation of the remainder</u>.

Substituting $F = \sum_{0}^{n} c_k z^k + R_n$ in (11) it is found

$$(1-z^2) \, R_n' = (z - \omega z^4) \, R_n + Q(z) \qquad (14)$$

where Q(z) is a certain polynomial of the following form

$$\alpha_0 z^n + \alpha_1 z^{n+1} + \alpha_2 z^{n+2} + \alpha_3 z^{n+3} + \alpha_4 z^{n+4}$$

the coefficients α_i are linear combinations of

$$c_{n+1}, \; c_{n+2}, \; c_{n+3}, \; c_{n+4}, \; c_{n+5}$$

The solution of (14) is, remembering $R_n(0) = 0$

$$R_n(z) = F(z) \int_0^z \frac{Q(\zeta)\,d\zeta}{F(\zeta)(1-\zeta^2)}$$

or $R_n(z) = F(z) \displaystyle\int^z \frac{\exp - \omega\left\{\zeta + \frac{\zeta^3}{3} + \frac{1}{2}\ln\frac{1-\zeta}{1+\zeta}\right\} Q(\zeta)\,d\zeta}{(1-\zeta^2)^{\frac{1}{2}}}$

As to the remainder E(n) in (12) the following can be proved:

$$\omega^r E(n,\omega) \to 0 \quad \text{for} \quad \omega \to \infty ,$$

r is a function of n increasing as $n \to \infty$.

This lemma may be proved by substituting the expression (15) in (13). This leads to tedious calculations and only the third term in the righthand member of (13) will be considered:

$$|R_n(t)| \leqq (1-t^2)^{-\frac{1}{2}} \int_0^t (1-u^2)^{-\frac{1}{2}} \exp \omega \left\{ t + \frac{t^3}{3} + \tfrac{1}{2} \ln \frac{1-t}{1+t} - u - \frac{u^3}{3} - \tfrac{1}{2} \ln \frac{1-u}{1+u} \right\} Q^*(u)du$$

$\psi(t) = t + \frac{t^3}{3} + \frac{1}{2} \ln \frac{1-t}{1+t}$ is monotonously decreasing and hence $\psi(t) - \psi(u) \leqq 0$.

From this it follows

$$|R_n(t)| \leqq (1-t^2)^{-\frac{1}{2}} \int_0^t (1-u^2)^{-\frac{1}{2}} Q^*(u)du$$

The coefficients of $Q^*(u)$ contain ω in the maximum power $\left[\frac{n+4}{5}\right]$ and consequently:

$$\frac{1}{\pi} \int_0^\infty e^{-\frac{\omega}{3} t^3} |R_n(t)| dt \to 0 \quad \text{for} \quad \omega \to \infty$$

with a certain power ω^{-r}, r increasing with n.

The final result is of course far from satisfactory, in the first place owing to the rough inequalities used in (13) and in the second place owing to the fact that the asymptotic series (12) oscillates very slowly, being a power series in $(\frac{\omega}{3})^{-\frac{1}{3}}$.

In the coefficient c_k ω occurs in a power $\leq \left[\frac{k}{5}\right]$. This means, that after five terms the essential gain in ω is only $\omega^{5/3} : \omega^{-1} = {}^{2/3}$

Therefore it is advisable to derive another asymptotic representation of $M_{\frac{\omega}{2},0}(z\omega)$. This may be performed by rearrangement of the series (12) according to equal powers $\omega^{-k/3}$ in such a way that apart from a sinus or cosinus factor the coefficients of $\omega^{-k/3}$ are constant numbers only depending of n.

The asymptotic series thus found is simply the Debije-series and may be obtained directly by means of Debije's method which uses a better contour in the definition integral (8).

Debije's method will be considered later on.

10. Rearrangement of the asymptotic development (12).

The content of this paragraph will be summarized briefly. The reader who is interested in the details may consult a paper about the same subject, to appear in applied scientific research.

The coefficient c_n is a polynomium in ω. c_n gives a contribution to these powers of $\omega^{-\frac{1}{3}}$ in the rearranged series for which the exponent e equals:

$$\begin{cases} e = n - 3 \left\{ \left[\frac{n}{5}\right] - 2j \right\} \leqq n & \begin{cases} n = 5l,\ 5l+2,\ 5l+4 \\ j = 0,1,2\ldots \end{cases} \\ e = n - 3 \left\{ \left[\frac{n}{5}\right] - 2j - 1 \right\} \leqq n & \begin{cases} n = 5l+1,\ 5l+3 \\ j = 0,1,2\ldots \end{cases} \end{cases}$$

In view of the sinus factor e can only have the following values

$$\begin{cases} e = 6p & p = 0,1,2\ldots \\ e = 6p + 4 \end{cases}$$

So we see that:

$$(\omega^{-})^{6p} \text{ gets contribution from } c_{6p},\ c_{6p+3},\ c_{6p+6}\ \ldots$$

$$(\omega^{-})^{6p+4}\ \text{"}\qquad\text{"}\qquad\text{"}\quad c_{6p+4},\ c_{6p+7}\ \ldots$$

Denoting the coefficient of ω^j in c_k by C_k^j we have from the recurrency relation:

$$\begin{cases} C_k^j = \frac{k-1}{k}\ C_{k-2}^j - \frac{1}{k}\ C_{k-5}^{j-1} \\ C_k^j = 0 \qquad j > \left[\frac{n}{5}\right] \end{cases} \tag{16}$$

Now the rearranged series can be written as follows:

$$M_{\frac{\omega}{2},0}(z\omega) \approx \frac{2^{\frac{1}{2}}\omega^{1/6}}{3^{1/6}\pi} \left\{ \Gamma\left(\tfrac{1}{3}\right) \cos\left(\tfrac{\omega}{2} - \tfrac{1}{6}\right)\pi \sum_0 a_p \omega^{-2p} + \right.$$

$$\left. + 2.3^{1/3}\, \Gamma\left(\tfrac{2}{3}\right) \cos\left(\tfrac{\omega}{2} + \tfrac{1}{6}\right)\pi \sum_0 b_p\, \omega^{-2p-\frac{4}{3}} \right\} \tag{17}$$

$$\begin{aligned} a_p &= \sum_{j=0}^{2p} 1.4.7 \ldots (6_p + 3j - 2)\ C_{6p+3j}^j \\ b_p &= \sum_{j=0}^{2p+2} 5.8.11 \ldots (6p + 3j + 2)\ C_{6p+3j+4}^j \end{aligned} \tag{18}$$

Writing $D(j,p) = 1.4.7 \ldots (6p+3j-2)\ C_{6p+3j}^j$ a rather complicated recurrency formula results viz.:

$$D(j,p) = \frac{(n-1)(n-3)(n-5)^2}{n(n-4)}\ D(j,p-1) - \frac{n-8}{n(n-4)(n-7)} \cdot$$

$$\left\{ (n-1)(n-3)(n-5)(n-7) + (n-1)(n-4)(n-5)(n-8) + (n-2)(n-4)(n-6)(n-8) \right.$$

$$D(j-1,p-1) + \frac{(n-8)(n-11)}{n(n-7)(n-10)} \left\{ (n-1)(n-5)(n-10) + (n-2)(n-6)(n-10) + \right.$$

$$\left. + (n-2)(n-7)(n-11) \right\} D(j-2,p-1) - \frac{(n-2)(n-8)(n-11)(n-14)}{n(n-10)} D(j-3,p\cdots$$

where $n = 6p + 3j$.

We have of course $D(j,p) = 0$ for $j > 3p$.

For $E(j,p) = 5.8.11 \ldots (6p + 3j + 2)\, C^{j}_{6p+3j+4}$

exactly the same relation holds; only n has now to be replaced by

$$n = 6p + 3j + 4$$

and $E(j,p) = 0 \quad$ for $j > 3p + 2$.

The initial D's are

$$D(0,0) = 1$$
$$D(0,1) = \frac{5}{4} \quad D(1,1) = -\frac{649}{70} \quad D(2,1) = \frac{54}{5} \quad D(3,1) = -\frac{364}{75}$$

Hence $a_0 = 1 \quad a_1 = -\frac{13}{900}$ (surprisingly small !) and analogously

$b_0 = -\frac{11}{280} = -0.0393$

$b_1 = 0.0144$.

Inserting these values in (17) we get:

$$M_{\frac{\omega}{2},0}(\omega) = \frac{2^{\frac{1}{2}}\,\omega^{1/6}}{3^{1/6}\,\pi} \left\{ \Gamma\left(\tfrac{1}{3}\right) \cos\left(\tfrac{\omega}{2} - \tfrac{1}{6}\right)\pi \left(1 - \frac{0.0144}{\omega^2} \ldots\right) + \right.$$
$$\left. + 2.3^{1/3}\, \Gamma\left(\tfrac{2}{3}\right) \cos\left(\tfrac{}{2} + \tfrac{1}{6}\right)\pi \left(-\frac{0.0393}{\omega^{4/3}} + \frac{0.0144}{\omega^{10/3}} \ldots\right) \right. \qquad (19)$$

As has been remarked (19) represents the Debije-series of $M_{\frac{\omega}{2},0}(2\omega)$ which will be derived afterwards in a more direct way.

11. From (19) a series may be obtained for the eigenvalues ω, i.e. the zero's of $M_{\frac{\omega}{2},0}(2\omega)$. The first approximation of $M_{\frac{\omega}{2},0}(2\omega)$ is:

$$M_{\frac{\omega}{2},0}(2\omega)=\frac{2^{\frac{1}{2}}\,\omega^{1/6}}{3^{1/6}\pi}\Gamma(1/3)\cos\left(\frac{\omega}{2}-\frac{1}{6}\right)\pi \quad + O(\omega^{-11/6})$$

From this we get as the first approximation of the eigenvalues:

$$\omega_l = 2l + 1 + 1/3 \qquad l = 0,1,2,3\ \ldots \tag{20}$$

From the convergent series representations of $M_{\frac{\omega}{2},0}(2\omega)$ the first eigenvalue is found to be $\omega_0 = 1,35217$.
This reveals the remarkable fact that (20) approximates the first eigenvalue $l = 0$ already very good. More accurate results may be obtained when assuming for ω_l a development of the following type

$$\begin{cases} \omega_l \approx v + \dfrac{2\alpha}{\pi v^{4/3}} + \dfrac{2\beta}{\pi v^{8/3}} + \ldots \\ v=2l+1+1/3 \end{cases} \tag{21}$$

(Of course a priori the exponents of the powers in v are unknown but soon it will be seen that $4/3$ and $8/3$ are the exact values).

Substituting (21) into (19) and equating powers of v we find for α and β :

$$\alpha = \frac{3^{5/6}.11.\,\Gamma(\frac{2}{3})}{280\,\Gamma(\frac{1}{3})}$$

$$\beta = -\frac{3^{7/6}.11^2.\Gamma^2(\frac{2}{3})}{280^2\,\Gamma^2\,(\frac{1}{3})}$$

hence:

$$\omega_l \approx v + 0{,}031580\, v^{-\frac{4}{3}} - 0{,}000904\, v^{-\frac{8}{3}} \ldots \tag{22}$$

$$\omega_1 = 1{,}35$$
$$\omega_2 = 3{,}340$$
$$\omega_3 = 5{,}3367$$
$$\omega_4 = 7{,}33554$$

12. <u>The exact derivation of the asymptotic series by means of Debije's method.</u>

Starting anew from the fundamental formula

$$M_{\frac{\omega}{2},0}(2\omega) = (2\omega)^{1/2}\,\frac{1}{2\pi i}\int^{(+1)} e^{\omega z}\left(\frac{z-1}{z+1}\right)^{\frac{\omega}{2}}\frac{dz}{(z^2-1)^{1/2}} \tag{1}$$

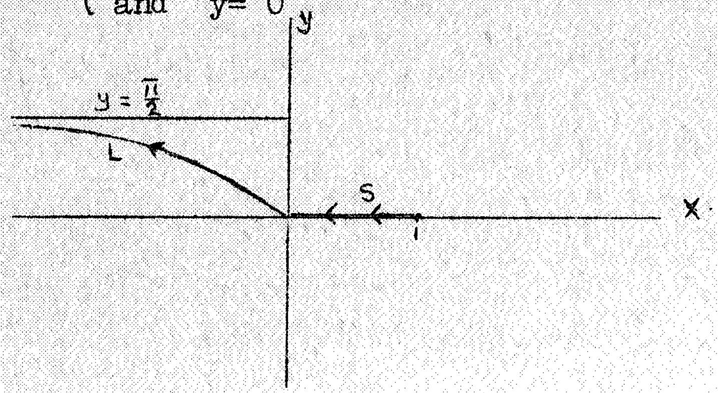
the asymptotic expansion will be derived by means of Debije's method of steepest descent.

A new contour is chosen as defined by

$$\text{Im}\,(z + \tfrac{1}{2}\ln\frac{z-1}{z+1}) = \text{constant}$$

or considering only the upper half plane:

$$\begin{cases} x^2 = 2y \operatorname{cotg} 2y + 1 - y^2 & 0 < y < \frac{\pi}{2} \\ \text{and} \quad y = 0 & \end{cases}$$



For (1) we write

$$M_{\frac{\omega}{2},0}(2\omega) = -\frac{(2\omega)^{\frac{1}{2}}}{\pi}\,\text{Re}\left[e^{\frac{\omega\pi i}{2}} \int_{S+L} e^{\omega\,\varphi(z)} \frac{dz}{(1-z^2)^{\frac{1}{2}}}\right] \qquad (2)$$

where $\varphi(z) = z + \frac{1}{2}\ln\frac{1-z}{1+z}$

S : a segment of the real axis (0,1) .

L : a curve in the second quadrant with asymptote $y = \frac{\pi}{2}$ and tangent $y = x \operatorname{tg}\frac{2\pi}{3}$ in the origin.

Now two ways may be followed:

A Perform the transformation $\zeta = \{-\varphi(z)\}^{\frac{1}{3}}$ and apply the Lagrange expansion :

$$\frac{1}{F'(z)\psi(z)} = \sum_{0}^{n-1} m_k \zeta^k + \frac{\zeta^n}{2\pi i}\int_C \frac{dt}{\psi(t)F^n(t)\{F(t)-\zeta\}} \qquad (3)$$

B Perform the substitution $v = \omega^{\frac{1}{3}} z$ and develop the integrand of of (2) as follows:

$$(1-z^2)^{-\frac{1}{2}} e^{\omega\,\varphi(z)} = e^{-\frac{v^3}{3}} \sum_{0} p_k(v)\,(\omega^{-\frac{2}{3}})^k$$

Both methods lead to different expressions for the remainder of the asymptotic series. Here we prefer method A. To calculate the coefficients of the asymptotic series method B is more suited.

Method B will be discussed in the next paragraph.

13. <u>Lemma</u>

$$(1-z^2)^{-\frac{1}{2}}\, e^{\omega(z+\frac{1}{2}\ln\frac{1-z}{1+z})} = e^{-\frac{v^3}{3}}\left\{\sum_{0}^{n-1} p_k(v)\,\omega^{-\frac{2k}{3}} + P_n\right\} \qquad (4)$$

$$v = \omega^{\frac{1}{3}} z$$

The polynomials $p_k(v)$ satisfy the recurrency formula:

$$\left\{ \begin{array}{l} p_{k+1}(v) = v^2\, p_k(v) - \int_0^v (\tau + \tau^4)\, p_k(\tau)\, d\tau \\ \qquad\qquad p_o = 1 \end{array} \right. \tag{5}$$

In particular

$$p_1 = \tfrac{1}{2}\, v^2 - \tfrac{1}{5}\, v^5$$
$$p_2 = \tfrac{3}{8}\, v^4 - \tfrac{17}{70}\, v^7 + \tfrac{1}{50}\, v^{10}$$

The remainder P_n may be represented in the following form:

$$R_n(v) = \omega^{-\frac{2n}{3}} (1 - v^2\, \omega^{-2/3})^{-1/2} \exp\left\{ v\omega^{2/3} + \frac{v^3}{3} + \frac{\omega}{2} \ln \frac{1 - v\omega^{-1/3}}{1 + v\omega^{-1/3}} \right\} \cdot$$
$$\cdot \int_0^{v\omega^{-1/3}} \frac{p_n'(\tau\, \omega^{1/3})\, d\tau}{(1-\tau^2)^{1/2} \exp \omega\{\tau + \frac{\tau^3}{3} + \frac{1}{2} \ln \frac{1-\tau}{1+\tau}\}} \tag{6}$$

<u>Proof.</u> Denoting $\sum_0^n p_k\, \omega^{-\frac{2k}{3}}$ by $F_n(v)$ we get:

$$\frac{\partial}{\partial v} \ln (F_n + P_n) = \frac{\partial}{\partial z} \left\{ -\frac{1}{2} \ln(1-z^2) + \omega(z + \frac{z^3}{3} + \frac{1}{2} \ln \frac{1-z}{1+z}) \right\} \cdot \frac{dz}{dv}$$

$$\frac{F_n' + P_n'}{F_n + P_n} = \frac{v - v^4}{\omega^{2/3} - v^2}$$

Formally it follows

$$(\omega^{2/3} - v^2) \sum_0^\infty p_k'\, \omega^{-\frac{2k}{3}} = (v - v^4) \sum_0^\infty p_k\, \omega^{-\frac{2k}{3}}$$

or

$$p_{k+1}'(v) = v^2\, p_k'(v) + (v - v^4)\, p_k(v) \tag{7}$$

Since $p_k(0) = 0$ $(k \geqslant 1)$ which follows from

$$1 = \sum_0^\infty p_k(0)\, \omega^{-\frac{2k}{3}}$$

we get integrating the last equality:

$$P_{k+1}(v) = \int_0^v \left\{ v^2\, p_k'(v) + (v - v^4)\, p_k(v) \right\} dv$$

and finally:

$$P_{k+1}(v) = v^2\, p_k(v) - \int_0^v (\tau + \tau^4)\, p_k(\tau)\, d\tau \tag{8}$$

From the definition $p_o(v) = 1$ and by means of (8) the next p_k's may be easily found.

For $R_n(v)$ the following differentialequation may be derived:

$$\omega^{2/3} R'_{n+1}(v) = v^2 R'_n(v) + (v - v^4) R_n(v)$$

or

$$(\omega^{2/3} - v^2) R'_n(v) + (v^4 - v) R_n(v) = \omega^{\frac{2(n-1)}{3}} p'_n(v)$$

A solution of the homogeneous equation is of course $R_0(v) = F(v) = \sum_0^{\infty} p_k \omega^{-\frac{2k}{3}}$

Hence the solution of the complete equation is:

$$R_n(v) = \omega^{\frac{2(n-1)}{3}} F(v) \int_0^v \frac{p'_n(\tau)\,d\tau}{F(\tau)(\omega^{2/3} - \tau^2)}$$

or

$$R_n(v) = \omega^{-\frac{2n}{3}} F(v) \int_0^{v\omega^{-1/3}} \frac{p_n(\tau\omega^{1/3})\,d\tau}{e^{\omega\left\{\tau + \frac{\tau^3}{3} + \frac{1}{2}\ln\frac{1-\tau}{1+\tau}\right\}}(1-\tau^2)^{1/2}} \qquad (9)$$

Formula (9) may be chosen as starting-point for the estimation of the remainder of the asymptotic series. However another method will be adopted in the following paragraphs.

Applying the lemma on (2) - the shape of the Debije contour is not essential here - it is found:

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{2^{1/2}\omega^{1/6}}{\pi} \operatorname{Re}\left[e^{\frac{\omega\pi i}{2}} \int_0^{\omega^{1/3}} e^{-\frac{v^3}{3}} \left\{ \sum_0^{n-1} p_k \omega^{-\frac{2k}{3}} + P_n \right\} dv - e^{\frac{\omega\pi i}{2}} \int_0^{\infty e^{\frac{2\pi i}{3}}} e^{-\frac{v^3}{3}} \left\{ \sum_0^{n-1} p_k \omega^{-\frac{2k}{3}} + P_n \right\} dv \right]$$

Since $p_k(v)$ is of the form v^{2k}. polynomial in v^3 this may be written as follows:

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{2^{1/2}\omega^{1/6}}{\pi} \left\{ \cos\frac{\omega\pi}{2} \cdot \sum_0^{n-1} \omega^{-\frac{2k}{3}} \int_0^{\infty} e^{-\frac{v^3}{3}} p_k(v)\,dv - \cos\left(\frac{\omega}{2} - \frac{2(k-1)}{3}\right)\pi \sum_0^{n-1} \omega^{-\frac{2k}{3}} \int_0^{\infty} e^{-\frac{v^3}{3}} p_k(v)\,dv \right\} + \text{Remainder}$$

$$= \frac{-(2\omega^{1/3})^{1/2}}{\pi}\, 2\sin\frac{k-1}{3}\pi \,.\, \cos\left(\frac{\omega}{2} - \frac{k}{3} - \frac{1}{6}\right)\pi \sum_0^{n-1} \omega^{-\frac{2k}{3}} \int_0^{\infty} e^{-\frac{v^3}{3}} p_k\,dv$$

$$= \frac{(6\omega^{1/3})^{1/2}}{\pi} \cos\left(\frac{\omega}{2} - \frac{1}{6}\right)\pi \sum_0^{\left[\frac{n-1}{3}\right]} \omega^{-2\ell} \int_0^{\infty} e^{-\frac{v^3}{3}} p_{3\ell}(v)\,dv + \text{Remainder}$$

$$+ \frac{(6\omega^{1/3})^{1/2}}{\pi} \cos\left(\frac{\omega}{2} + \frac{1}{6}\right)\pi \sum_0^{\left[\frac{n-3}{3}\right]} \omega^{-2\ell-} \int_0^{\infty} e^{-\frac{v^3}{3}} p_{3\ell+2}(v)\,dv + \text{Rem.}$$

Writing $\mu_k = \int_0^\infty e^{-\frac{v^3}{3}} p_k(v)\, dv$

this result may be written

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{6^{\frac{1}{2}}\omega^{\frac{1}{6}}}{\pi} \cos\left(\frac{\omega}{2} - \frac{1}{6}\right)\pi \sum_{0}^{\left[\frac{n-1}{3}\right]} \mu_{3\ell}\,\omega^{-2\ell} + \tag{10}$$
$$+ \frac{6^{\frac{1}{2}}\omega^{\frac{1}{6}}}{\pi} \cos\left(\frac{\omega}{2} + \frac{1}{6}\right)\pi \sum_{0}^{\left[\frac{n}{3}\right]-1} \mu_{3\ell+2}\,\omega^{-2\ell-\frac{2}{3}} + C_n$$

where

$$C_n = \frac{2^{\frac{1}{2}}\omega^{\frac{1}{6}}}{\pi} \operatorname{Re}\left[e^{\frac{\omega\pi i}{2}} \int_{\omega^{1/3}}^{\infty} e^{-\frac{v^3}{3}} \sum_{0}^{n-1} p_k(v)\,\omega^{-\frac{2k}{3}}\, dv \right] +$$
$$+ \frac{2^{\frac{1}{2}}\omega^{\frac{1}{6}}}{\pi} \operatorname{Re}\left[e^{\frac{\omega\pi i}{2}} \int_{0}^{\omega^{1/3}} e^{-\frac{v^3}{3}} P_n\, dv - e^{\frac{\omega\pi i}{2}} \int_{0}^{\infty e^{\frac{2\pi i}{3}}} e^{-\frac{v^3}{3}} P_n\, dv \right]$$

The reader will have no difficulty to prove that $C_n = O(\omega^{-\frac{2(n+1)}{3}})$.
However the remainder occurring in the A-method will be seen to be more suited for the calculation of an approximation.

From (5) it follows that p_k is of the form

$$p_k(v) = v^{2k} \sum_{j=0}^{k} a_{kj}\, v^{3j}$$

Hence

$$\mu_k = \sum_{0}^{k} a_{kj} \int_0^\infty e^{-\frac{v^3}{3}} v^{2k+3j}\, dv =$$
$$= \sum_{0}^{k} 3^{\frac{2k+3j-2}{3}}\, \Gamma\left(\frac{2k+3j+1}{3}\right) a_{kj} = 3^{\frac{2k-2}{3}}\, \Gamma\left(\frac{2k+1}{3}\right)$$
$$\sum_{0}^{k} (2k+3j-2)(2k+3j-5)\ldots(2k+1)\, a_{kj}$$

In particular

$$\mu_0 = 3^{-2/3}\,\Gamma\left(\tfrac{1}{3}\right) \qquad \mu_2 = 3^{2/3}\,\Gamma\left(\tfrac{5}{3}\right)\left\{ a_{20} + 5a_{21} + 40\, a_{22} \right\} =$$
$$= -\frac{11}{280}\, 3^{2/3}\,\Gamma\left(\tfrac{5}{3}\right)$$

The actual computation of the higher μ_k's remains however as complicated as in our former considerations. Moreover we meet the same difficulty, that there seems to be no a priori reason why the μ_k's become so small !

§ 14. The original method of Debije by means of the Lagrange-Bürmann expansion :

Starting again from (2) we find:

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{(2\omega)^{1/2}}{\pi} \quad \mathrm{Re}\left[e^{\frac{\omega\pi i}{2}} \int_0^{\infty} e^{-\omega\xi^3} \frac{d\xi}{\frac{d\xi}{dz}(1-z^2)^{1/2}} - \right.$$

$$\left. - e^{\frac{\omega\pi i}{2}} \int_0^{\infty e^{\frac{2\pi i}{3}}} e^{-\omega\xi^3} \frac{d\xi}{\frac{d\xi}{dz}(1-z^2)^{1/2}} \right]$$

According to the Lagrange expansion we have

$$\frac{1}{\frac{d\xi}{dz}(1-z^2)^{1/2}} = \sum_0^{n-1} m_k \xi^k + R_n(\xi)$$

where $m_k = \frac{1}{2\pi i} \oint \frac{dt}{(1-t^2)^{1/2}(-t-\frac{1}{2}\ln\frac{1-t}{1+t})}$

(integration round the origin)

$$R_n(\xi) = \frac{1}{2\pi i} \int_C \frac{dt}{(1-t^2)^{1/2}(-t-\frac{1}{2}\ln\frac{1-t}{1+t})^{n/3}\left\{(-t-\frac{1}{2}\ln\frac{1-t}{1+t})^{1/3} - \xi\right\}}$$

path of integration a curve enclosing the Debije contour i.e. the points where $(-t-\frac{1}{2}\ln\frac{1-t}{1+t})^{1/3} = \xi$.
Proceeding along the lines we find:

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{(2\omega)^{1/2}}{3\pi} \sum_0^{n-1} 2\, m_k\, \Gamma(\tfrac{k+1}{3}) \cdot \sin\tfrac{k+1}{3}\pi \cdot \sin(\tfrac{\omega}{2} + \tfrac{k+1}{3})\pi \cdot \omega^{-\frac{k+1}{3}} +$$

$$+ \text{ remainder} \qquad (11)$$

$$\text{remainder} = \frac{(2\omega)^{1/2}}{\pi} \left\{ \cos\frac{\omega\pi}{2} \int_0^{\infty} e^{-\omega\xi^3} \xi^n R_n(\xi) d\xi - \right.$$

$$\left. - \mathrm{Re}\left[e^{\frac{\omega\pi i}{2}} \int^{\infty e^{\frac{2\pi i}{3}}} e^{-\omega\xi^3} \xi^n R_n(\xi) d\xi \right] \right. \qquad (12)$$

It is easily seen, that

$$m_k \neq 0 \text{ only if } \begin{array}{l} k = 6p \\ k = 6p + 4 \end{array} \qquad p = 0,1,2\ldots$$

This means, that (11 may be written in the following form:

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{2^{1/2}\omega^{1/6}}{3^{1/2}\pi} \cos(\tfrac{\omega}{2} - \tfrac{1}{6})\pi \sum_0^{n-1} m_{6p}\Gamma(2p + \tfrac{1}{3})\,\omega^{-2p} +$$

$$+ \frac{2^{1/2}\omega^{1/6}}{3^{1/2}\pi} \cos(\tfrac{\omega}{2} + \tfrac{1}{6})\pi \sum_0^{n-1} m_{6p+4}\,\Gamma(2p + \tfrac{5}{3})\,\omega^{-2p-2/3} + \text{Remainder.} \qquad (13)$$

Comparation of this result with (10) shows, that

$$\mu_{3\ell} = \tfrac{1}{3}\Gamma(2p + \tfrac{1}{3})\, m_{6p} \; ; \quad \mu_{3\ell+2} = \tfrac{1}{3}\Gamma(2p + \tfrac{5}{3})\, m$$

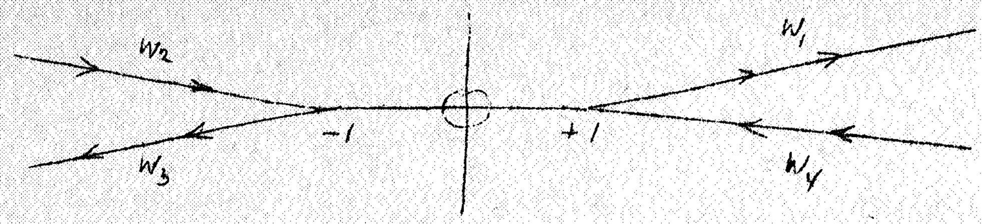
CONFLUENT HYPERGEOMETRIC FUNCTIONS

door H.A. Lauwerier.

We start from the formula:

$$m_k = \frac{1}{2\pi i} \oint \frac{dt}{(1-t^2)^{1/2}\left(-t - \frac{1}{2}\ln\frac{1-t}{1+t}\right)^{\frac{k+1}{3}}} \quad \cdots (14)$$

It is advisable to choose a contour on which $\left|-t - \frac{1}{2}\ln\frac{1-t}{1+t}\right|$ is large. Now this factor is minimal in the origin, so t has to be taken as far as possible from 0. Therefore in view of the branchpoints ± 1 the following path is chosen:



1 w_1 along the positive real axis from 1 to ∞, $\arg(1-t^2) = -\pi$

2 a large semicircle round 0 with indefinitely increasing radius.

3 w_2 from $-\infty$ to -1, $\arg(1-t^2) = \pi$

4,5,6 in a similar way in the lower halfplane.

The contributions of the semicircles tend to zero. Remembering that m_k is real, (14) may be transformed into:

$$m_k = \frac{1}{\pi} \mathrm{Re}\left\{\left(e^{\frac{k+1}{3}\pi i} - 1\right)\int_1^\infty \frac{dt}{(t^2-1)^{1/2}\left\{t + \frac{1}{2}\ln\frac{t-1}{t+1} + \frac{\pi i}{2}\right\}}\right\} \quad \cdots (15)$$

To get a estimation of (m_k) we remark that for small t values the factor $\left\{\left(t + \frac{1}{2}\ln\frac{t-1}{t+1}\right)^2 + \frac{\pi^2}{4}\right\}$ can be minorated by $\frac{\pi^2}{4}$. When $\left(t + \frac{1}{2}\ln\frac{t-1}{t+1}\right)$ has passed his zero value we make use of a better approximation:

$$1 < t < a \qquad \left(t + \tfrac{1}{2}\ln\frac{t-1}{t+1}\right)^2 + \frac{\pi^2}{4} \geqslant \frac{\pi^2}{4}$$

$$t > a \qquad \left(t + \tfrac{1}{2}\ln\frac{t-1}{t+1}\right)^2 + \frac{\pi^2}{4} \geqslant \left(t - \tfrac{1}{2}\ln\frac{a+1}{a-1}\right)^2 + \frac{\pi^2}{4}$$

Writing $b = \frac{1}{2}\ln\frac{a+1}{a-1}$ we get

$$|m_k| < \frac{5}{\pi}\left\{\left(\frac{2}{\pi}\right)^{\frac{k+1}{3}}\int_1^a \frac{dt}{(t^2-1)^{1/2}} + \int_a^\infty \frac{dt}{\left\{(t-b)^2+\frac{\pi^2}{4}\right\}^{\frac{k+1}{6}}(t^2-1)^{1/2}}\right\} \tag{16}$$

$$\begin{cases} s=1 & \text{for } k = 6p\,,\ k = 6p+4 \\ s=2 & \text{for } k = 6p+2 \end{cases} \qquad \int_1^a \frac{dt}{(t^2-1)^{1/2}} = \ln(a+\sqrt{a^2-1})$$

$$\int_a^\infty \frac{dt}{\left\{(t-b)^2+\frac{\pi^2}{4}\right\}^{\frac{k+1}{6}}(t^2-1)^{1/2}} = \left(\frac{2}{\pi}\right)^{\frac{k+1}{3}}\int_{\frac{2}{\pi}(a-b)}^\infty \frac{du}{(u^2+1)^{\frac{k+1}{6}}\left\{u^2+\frac{4}{\pi}bu-\frac{4}{\pi^2}(b^2-1)\right\}^{1/2}}$$

<u>lemma</u>: for $a > \frac{5}{4}$ we have

$$u^2+\frac{4}{\pi}bu-\frac{4}{\pi^2}(b^2-1) > u^2.$$

Hence:

$$\int_0^\infty \frac{dt}{\left\{(t-b)^2+\frac{\pi^2}{4}\right\}^{\frac{k+1}{6}}(t^2-1)^{\frac{1}{2}}} < \left(\frac{2}{\pi}\right)^{\frac{k+1}{3}}\int_{\frac{\pi}{2}(a-b)}^\infty \frac{du}{(u^2+1)^{\frac{k+1}{6}}u}$$

<u>lemma</u>

$$\int_0^\infty \frac{du}{u(u^2+1)^{\frac{k+1}{6}}} \leq \frac{3}{(k+1)c^2(1+c^2)^{\frac{k-5}{6}}}$$

Inserting this in (16) we get finally:

$$|m_k| < \frac{s}{\pi}\left(\frac{2}{\pi}\right)^{\frac{k+1}{3}}\left\{\ln(a+\sqrt{a^2-1}) + \frac{3}{(k+1)c^2(1+c^2)^{\frac{k-5}{6}}}\right\} \tag{17}$$
$$\begin{cases} c = \frac{2}{\pi}\left(a-\frac{1}{2}\ln\frac{a+1}{a-1}\right) \\ a > 5/4 \end{cases}$$

Taking c = 1 or a = 1,938 a useful formula may be obtained:

$$|m_{2k}| < s\left(\frac{2}{\pi}\right)^{\frac{2k+4}{3}}\left\{0{,}64 + \frac{3}{(2k+1)\,2^{\frac{2k+1}{6}}}\right\} \tag{18}$$

§ 17. <u>Estimation of the remainder of the asymptotic series</u>. The remainder of the Lagrange-Bürmann expansion used in the derivation of the asymptotic series has the form:

$$R_n(\zeta) = \frac{1}{2\pi i}\int_C \frac{dt}{(1-t^2)^{1/2}\left(-t-\frac{1}{2}\ln\frac{1-t}{1+t}\right)^{n/3}\left\{\left(-t-\frac{1}{2}\ln\frac{1-t}{1+t}\right)^{1/3}-\zeta\right\}} \tag{19}$$

a/ $\arg\zeta = 0$

b/ arg $\zeta = 2\pi$

The contour of § 14 may be used here with success, and we find easily, denoting $t + \frac{1}{2}\ln\frac{1-t}{1+t}$ by θ :

$$|R_1(\zeta)| \leqslant \frac{1}{\pi}\int_1^\infty \frac{T\,dt}{(t^2-1)^{1/2}\left(\theta^2+\frac{\pi^2}{4}\right)^{1/6}\left|\left(\theta+\frac{\pi i}{2}\right)^{1/3}-\zeta e^{\frac{\pi i}{3}}\right|\left|\left(\theta+\frac{\pi i}{2}\right)^{1/3}-\zeta\right|} \tag{20}$$

1 $n = 6p+4$, the asymptotic series ends with coefficient m_{6p}.

$$T = \left(\theta^2+\frac{\pi^2}{4}\right)^{1/6} + 3^{1/2}|\zeta|$$

2 $n = 6p+6$, the asymptotic series ends with coefficient m_{6p+4}.

$$T = \left(\theta^2+\frac{\pi^2}{4}\right)^{1/6}$$

Estimation of the product

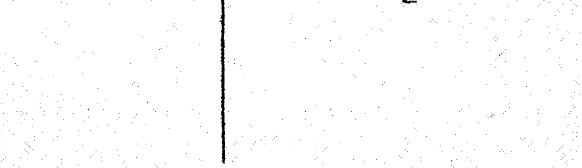
$$\left|\left(\theta+\frac{\pi i}{2}\right)^{1/3}-\zeta e^{\frac{\pi i}{3}}\right|\left|\left(\theta+\frac{\pi i}{2}\right)^{1/3}-\zeta\right|$$

1 $\arg\zeta = 0$

The path of $\left(\theta+\frac{\pi i}{2}\right)^{\frac{1}{3}}$ is a curve F with equation

$$3x^2y - y^3 = \frac{\pi}{2}$$

$\zeta e^{\frac{\pi i}{3}}$ and ζ represent points Z_1 , Z_2 of two fixed halflines. If P is a variable point of F our problem is reduced to that of finding a minorant of the product $PZ_1 \cdot PZ_2$



Choosing the axis of symmetry of F as new X-axis we get better symmetrical relations. Then new variables t, u, z

$$\begin{cases} x = \left(\frac{\pi}{2}\right)^{1/3} t \\ y = \left(\frac{\pi}{2}\right)^{1/3} \frac{u(t-1)}{3^{1/2}} \\ \zeta = \left(\frac{\pi}{2}\right)^{1/3} \frac{2z}{3^{1/2}} \end{cases}$$

will be introduced and it is found:

$$PF_1^2 \cdot PF_2^2 = \frac{\left(\frac{\pi}{2}\right)^{4/3}}{9t^2}\left\{9t^2(t-z)^4 + 6t(t-z)^2(t^3+tz^2-1) + (t^3-tz^2-1)^2\right.$$

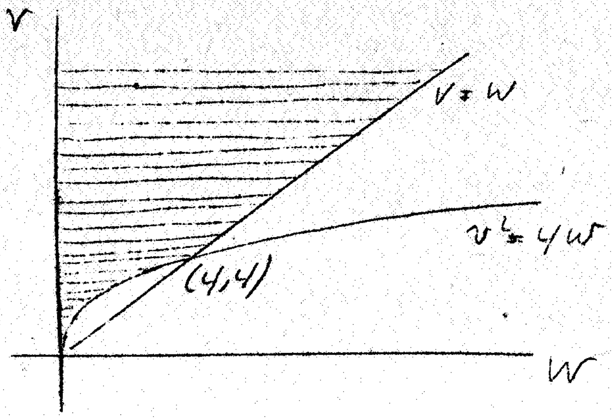
Introducing again new variables v, w defined by

$$v = 4t^2(t-z)$$
$$w = 4t(t-z)^2$$

we get

$$PF_1 \cdot PF_2 = \frac{(\pi/2)^{2/3}}{3t}\left(v^2 - vw + w^2 - v - w + 1\right)^{1/2} \quad . \quad . \quad . \quad (21)$$

$$v^2 \geq 4w \geq 0 \ , \quad v(v-w) \geq 0$$



<u>lemma</u> There exists a constant a with the property that for all admissible v, w the following inequality holds:

$$v^2 - vw + w^2 - v - w + 1 \geq \frac{9v^2}{\left\{a(v-w) + (4vw)^{1/3}\right\}^2}$$

<u>proof</u> tedious calculations. It appears that $a \geq 2 + 4 \cdot 3^{-\frac{1}{2}} = 4{,}309$.
From this it follows directly

$$PF_1 \cdot PF_2 \geq \frac{(\pi/2)^{2/3}}{1 + az}$$

or

$$PF_1 \cdot PF_2 \geq \frac{(\pi/2)^{2/3}}{1 + b\zeta} \qquad (22)$$

$$b = \left(2^{1/3}3^{1/2} + 2^{4/3}\right) \cdot \pi^{-1/3} = 3{,}211$$

2° $\arg \zeta = \frac{2\pi}{3}$

From the geometrical interpretation we read immediately

$$\left|\left(\theta + \frac{\pi i}{2}\right)^{1/3} - \zeta e^{\frac{\pi i}{3}}\right| \left|\left(\theta + \frac{\pi i}{2}\right)^{1/3} - \zeta\right| \geqslant OQ^2 = \left(\frac{\pi}{2}\right)^{2/3}$$

A simple calculation shows that even the following stronger inequality holds:

$$\left|\left(\theta + \frac{\pi i}{2}\right)^{1/3} - \zeta e^{\frac{\pi i}{3}}\right| \left|\left(\theta + \frac{\pi i}{2}\right)^{1/3} - \zeta\right| \geqslant \left|\theta + \frac{\pi i}{2}\right|^{1/3} \qquad (23)$$

Denoting the right-hand member of (18) by e_k :

$$e_k = \left(\frac{2}{\pi}\right)^{\frac{k+4}{3}} \left\{ 0{,}64 + \frac{3}{(k+1)\, 2^{\frac{k+1}{6}}} \right\}$$

we get for $|R_n(\zeta)|$ the following estimation:

1° $\arg \zeta = 0$

$$|R_n(\zeta)| \leqslant \left(\frac{2}{\pi}\right)^{2/3} (1 + b\zeta) \cdot \frac{1}{\pi} \int_1^\infty \frac{T}{(t^2-1)^{1/2} \left(\theta^2 + \frac{\pi^2}{4}\right)^{3/6}}$$

a $n = 6p+4$

$$|R_n(\zeta)| \leqslant \left(\frac{2}{\pi}\right)^{2/3} (1 + b\zeta) \left\{ e_{6p+2} + 3^{1/2} \zeta\, e_{6p+3} \right\}$$

$$\leqslant \left(\frac{2}{\pi}\right)^{2/3} (1 + b\zeta)(1 + c\zeta)\, e_{6p+2} \qquad (24)$$

$$b = \left(\frac{2}{\pi}\right)^{1/3} (2 + \sqrt{3}) \quad , \quad c = \left(\frac{2}{\pi}\right)^{1/3} \sqrt{3} .$$

b $n = 6p+6$

$$|R_n(\zeta)| \leqslant \left(\frac{2}{\pi}\right)^{2/3} (1 + b\zeta)\, e_{6p+4} \qquad (25)$$

2° $\arg \zeta = \frac{2\pi}{3}$

a $$|R_n(\zeta)| \leqslant \left(\frac{2}{\pi}\right)^{2/3} (1 + c\zeta)\, e_{6p+4} \qquad (26)$$

b $$|R_n(\zeta)| \leqslant \left(\frac{2}{\pi}\right)^{2/3} e_{6p+6} \qquad (27)$$

For the remainder of the asymptotic series we found the expression:

$$|C_n| \leqslant \frac{(2\omega)^{1/2} \left|\cos \frac{\omega\pi}{2}\right|}{\pi} \int_0^\infty e^{-\omega\zeta^3} \zeta^n |R_n(\zeta)|\, d\zeta +$$

$$+ \frac{(2\omega)^{1/2}}{\pi} \int_0^\infty e^{-\omega\zeta^3} \zeta^n \left|R_n\left(\zeta e^{\frac{2\pi i}{3}}\right)\right| d\zeta \qquad (28)$$

The inequalities (24) (25) (26) (27) can be substituted in (28) and we get integrals of type

$$\int_0^\infty e^{-\omega\zeta^3}\zeta^m d\zeta = \frac{1}{3}\,\omega^{-\frac{m+1}{3}}\,\Gamma\left(\frac{m+1}{3}\right)$$

The final result may be expressed as follows:

Summary

$M_{\frac{\omega}{2},0}(2\omega)$ admits for real positive values of ω the asymptotic representation

$$M_{\frac{\omega}{2},0}(2\omega) = \frac{2^{3/2}\omega^{1/2}}{3\pi}\sum_0^{n-1} m_k\,\Gamma\left(\frac{k+1}{3}\right)\sin\frac{k+1}{3}\pi\,\sin\left(\frac{\omega}{2}+\frac{k+1}{3}\right)\pi\cdot\omega^{-\frac{k+1}{3}} + C_n$$

$m_k \neq 0$ only if k = even

$m_0 = 3^{1/3}$ $\quad m_4 = -\frac{11}{280}\cdot 3^{5/3}$ $\quad m_6 = -\frac{13}{400}\cdot 3^{1/3}$

$$|m_k| \leqslant e_k = s\left(\frac{2}{\pi}\right)^{\frac{k+4}{3}}\left\{0{,}64 + \frac{3}{(k+1)\,2^{\frac{k+1}{6}}}\right\}$$

s = 1 if k = $6p$, $6p+4$

s = 2 if k = $6p+2$

1° The finite series stops at the term k = $6p$ taking n = $6p+4$ we get

$$|C_n| \leqslant \frac{2^{7/6}\omega^{1/2}}{3\pi^{5/3}}\left\{\left|\cos\frac{\omega\pi}{2}\right|(\gamma_0+c_1\gamma_1+c_2\gamma_2)e_{n-2} + (\gamma_0+c_3\gamma_1)e_n\right\}$$

2° id. at k = $6p+4$. Take n = $6p+6$

$$|C_n| \leqslant \frac{2^{7/6}\omega^{1/2}}{3\pi^{5/3}}\left\{\left|\cos\frac{\omega\pi}{2}\right|(\gamma_0+c_4\gamma_1)e_{n-2} + \gamma_0 e_n\right\}$$

$$\gamma_j = \Gamma\left(\frac{n+j+1}{3}\right)\omega^{-\frac{n+j+1}{3}}$$

$c_1 = 4{,}70$ $\quad c_2 = 4{,}78$ $\quad c_3 = 1{,}49$ $\quad c_4 = 3{,}21$

Numerical exemple

$\omega = 8$

$M_{4,0}(8) = 1{,}4494 - 0{,}0136 - 0{,}0013 + C_{10} = 1{,}4345 + C_{10}$

$C_{10} \leqslant 0{,}0009$

$m_6 = -\ 0{,}047$

$e_6 = 0{,}184$